

The mean number of 3-torsion elements in ray class groups of quadratic fields

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Abstract

We determine the average number of 3-torsion elements in the ray class groups of fixed (integral) conductor c of quadratic fields ordered by absolute discriminant, generalizing Davenport and Heilbronn's theorem on class groups. A consequence of this result is that a positive proportion of such ray class groups of quadratic fields have trivial 3-torsion subgroup whenever the conductor c is taken to be a squarefree integer having very few prime factors none of which are congruent to 1 mod 3. Additionally, we compute the second main term for the number of 3-torsion elements in ray class groups with fixed conductor of quadratic fields with bounded discriminant.

1 Introduction

In 1971, Davenport and Heilbronn [7] determined that the mean number of 3-torsion elements in the class groups of real (resp. imaginary) quadratic fields ordered by absolute discriminant is $\frac{4}{3}$ (resp. 2). In this paper, we determine the average number of 3-torsion elements in *ray* class groups of fixed integral conductor of quadratic fields ordered by their discriminant. More precisely, we prove the following theorem.

Theorem 1. *For any integer c , let m denote the number of distinct primes $p \mid c$ such that $p \equiv 1 \pmod{3}$. When quadratic fields are ordered by absolute discriminant:*

- (a) *The average number of 3-torsion elements in the ray class groups of conductor c of real quadratic fields is:*

$$\begin{cases} 3^m \cdot \left(1 + \frac{1}{3} \cdot \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 3 \nmid c, \\ 3^m \cdot \left(1 + \frac{2}{7} \cdot \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 3 \parallel c, \text{ and} \\ 3^{m+1} \cdot \left(1 + \frac{5}{7} \cdot \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 9 \mid c. \end{cases}$$

- (b) *The average number of 3-torsion elements in the ray class groups of conductor c of imaginary quadratic fields is:*

$$\begin{cases} 3^m \cdot \left(1 + \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 3 \nmid c, \\ 3^m \cdot \left(1 + \frac{6}{7} \cdot \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 3 \parallel c, \text{ and} \\ 3^{m+1} \cdot \left(1 + \frac{15}{7} \cdot \left(\prod_{p \mid c} 1 + \frac{p}{p+1}\right)\right) & \text{if } 9 \mid c. \end{cases}$$

Note that in the formulas given in Theorem 1, the case $c = 1$ recovers Davenport-Heilbronn's theorem on the average number of 3-torsion elements in the class groups of real and imaginary quadratic fields [7, Theorem 3]. The Cohen-Lenstra heuristics on asymptotics of p -torsion in class groups of quadratic fields were inspired by the constants appearing in [7] as well as computations. It would be interesting to have analogous heuristics for ray class groups that would explain the constants appearing in Theorem 1.

While the above mean values do depend on the conductor c , if we instead average over quadratic fields with discriminant coprime to the conductor, we obtain constants that only depend on the number of primes dividing c .

Theorem 2. *Fix a positive integer c with n distinct prime factors, and let m denote the number of distinct primes $p \mid c$ that are congruent to 1 mod 3. When quadratic fields are ordered by their absolute discriminant:*

- (a) *The average number of 3-torsion elements in the ray class groups of conductor c of real quadratic fields with discriminant coprime to c is:*

$$\begin{cases} 3^m \cdot \left(1 + \frac{2^n}{3}\right) & \text{if } 3 \nmid c, \\ 3^m \cdot \left(1 + \frac{2^{n-1}}{3}\right) & \text{if } 3 \parallel c, \text{ and} \\ 3^{m+1} \cdot (1 + 2^{n-1}) & \text{if } 9 \mid c. \end{cases}$$

- (b) *The average number of 3-torsion elements in the ray class groups of conductor c of imaginary quadratic fields with discriminant coprime to c is:*

$$\begin{cases} 3^m \cdot (1 + 2^n) & \text{if } 3 \nmid c, \\ 3^m \cdot (1 + 2^{n-1}) & \text{if } 3 \parallel c, \text{ and} \\ 3^{m+1} \cdot (1 + 3 \cdot 2^{n-1}) & \text{if } 9 \mid c. \end{cases}$$

The formulas in Theorem 1 also imply that a positive proportion of quadratic fields have trivial 3-torsion subgroups in their ray class groups for certain conductors c . Note that if p is a prime congruent to 1 mod 3, then there are non-trivial 3-torsion elements in the ray class group of \mathbb{Q} of conductor p , and so the size of the ray class group of any conductor divisible by p of any quadratic field is strictly greater than 1. (A similar statement is true for conductor 9.) However, if c is a product of a small enough number of primes all of which are not congruent to 1 mod 3 and $9 \nmid c$, then we have the following corollary of Theorem 1, which says that there are infinitely many ray class groups of such conductor of quadratic fields with trivial 3-torsion subgroup:

Corollary 3. (a) *If c is a product of 1 or 2 distinct primes each congruent to 2 mod 3, then a positive proportion of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c . A positive proportion of real quadratic fields also have trivial 3-torsion subgroup in their ray class groups of conductor $3c$.*

- (b) *If p is a prime congruent to 2 mod 3, then over 50% of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor p . Additionally, over 50% of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor $3p$.*

- (b) *If c is equal to 3 or a prime congruent to 2 mod 3, then a positive proportion of imaginary quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c .*

Finally, we may combine our methods with those of [2] to compute the second main term for the mean number of 3-torsion elements in ray class groups of quadratic fields ordered by absolute discriminant. More

precisely, we prove the following refinement of Theorem 1. For a quadratic field K_2 , let $\text{Cl}_3(K_2, c)$ denote the 3-torsion subgroup of the ray class group of K_2 of conductor c .

Theorem 4. *For any positive integer c coprime to 3, let m denote the number of distinct primes dividing c that are congruent to 1 mod 3. When quadratic fields are ordered by absolute discriminant:*

$$\begin{aligned} \sum_{0 < \text{Disc}(K_2) < X} \# \text{Cl}_3(K_2, c) &= 3^m \cdot \left(1 + \frac{1}{3} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \cdot \sum_{0 < \text{Disc}(K_2) < X} 1 \right. \\ &\quad + \frac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{15\Gamma(2/3)\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)} \right) \cdot \prod_{p|c} \left(1 + \frac{p(1 - p^{1/3})}{1 - \frac{p(p+1)}{p^{1/3}+1}} \right) \cdot X^{5/6} \Bigg) \\ &\quad + O_\epsilon(X^{5/6-1/48+\epsilon}), \text{ and} \\ \sum_{-X < \text{Disc}(K_2) < 0} \# \text{Cl}_3(K_2, c) &= 3^m \cdot \left(1 + \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \cdot \sum_{-X < \text{Disc}(K_2) < 0} 1 \right. \\ &\quad + \frac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{5\Gamma(2/3)\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)} \right) \cdot \prod_{p|c} \left(1 + \frac{p(1 - p^{1/3})}{1 - \frac{p(p+1)}{p^{1/3}+1}} \right) \cdot X^{5/6} \Bigg) \\ &\quad + O_\epsilon(X^{5/6-1/48+\epsilon}). \end{aligned}$$

To derive the result for 3-torsion ideal classes in class groups of quadratic fields, Davenport and Heilbronn first provide asymptotic formulae for the number of cubic fields having bounded discriminant and sieve to count the *nowhere totally ramified* cubic fields. These are degree 3 extensions K_3 of \mathbb{Q} in which any rational prime p that ramifies is of the form $(p) = \mathfrak{p}_1^2 \mathfrak{p}_2$ where \mathfrak{p}_1 and \mathfrak{p}_2 are two distinct primes of K_3 . They prove that the number of nowhere totally ramified cubic fields having bounded discriminant determines the number of 3-torsion ideal classes in quadratic fields with the same bound on their discriminant, and so they deduce the above theorem.

To prove Theorems 1 and 2, we combine a generalization of Davenport and Heilbronn's results on asymptotics for cubic fields given in [2] with a parametrization of the number of 3-torsion elements of ray class groups of quadratic fields with bounded discriminant in terms of the number of appropriate (pairs of) cubic fields with related bounds on their discriminants. We now outline the rest of the article.

In Section 2, we fix an integral conductor c and a quadratic field K_2 , and we compare the number of 3-torsion ideal classes in the ray class group of K_2 of conductor c to the number of pairs of cubic fields whose discriminants satisfy certain c^2 -divisibility conditions (see Theorem 2.4). We additionally study the action of $\text{Gal}(K_2/\mathbb{Q})$ on this 3-torsion subgroup in order to relate the number of 3-torsion ideal classes with a fixed action of $\text{Gal}(K_2/\mathbb{Q})$ to certain singleton cubic fields whose discriminants satisfy similar c^2 -divisibility conditions. In Section 3, we utilize a strengthening of Davenport-Heilbronn's theorem given in [2] that computes the density of discriminants of cubic fields satisfying certain *acceptable* local conditions. We are then able to prove Theorems 1 and 2 as well as Corollary 3 in Section 4. Finally, in Section 5 we prove Theorem 4 by computing the second main term for the average number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields with bounded discriminant, building on work of [2].

2 Parametrization of 3-torsion elements in ray class groups of quadratic fields

We begin by describing a bijection between index-3 subgroups of ray class groups of quadratic fields and certain pairs of cubic fields. This will allow us to determine the number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields using a generalization given in [2] of Davenport-Heilbronn's asymptotic formulae on the density of discriminants of cubic fields.

2.1 Ray class groups and fields

First, we recall the definition of the ray class group of a number field K . Because we will eventually range over all quadratic fields, we only consider ray class groups whose finite part of the modulus is integral (so that it can be fixed independently of the quadratic field). Additionally, because ramification at infinity only affects the size of the 2-torsion subgroup in the (narrow) ray class groups, we work with ray class groups with trivial infinite part of the modulus. Under these restrictions, we refer to the rational positive generator of the modulus as the *conductor*.

Fix $c \in \mathbb{Z}$, and let $\mathcal{I}_c(\mathcal{O}_K)$ denote the subgroup of fractional ideals of \mathcal{O}_K generated by prime ideals coprime to $c\mathcal{O}_K$. Additionally, let $\mathcal{P}_{1,c}(\mathcal{O}_K)$ denote the subgroup of principal ideals (α) such that $\alpha \equiv 1 \pmod{c\mathcal{O}_K}$. We then define the *ray class group of conductor c* as the quotient

$$\mathrm{Cl}(K, c) := \mathcal{I}_c(\mathcal{O}_K) / \mathcal{P}_{1,c}(\mathcal{O}_K). \quad (1)$$

In this notation, the ideal class group of a field K is denoted $\mathrm{Cl}(K, 1)$. Additionally, let $\mathrm{Cl}_p(K, c)$ denote the p -torsion subgroup of $\mathrm{Cl}(K, c)$ for any prime p .

There is another (equivalent) definition of $\mathrm{Cl}(K, c)$ as a quotient of the ideles. More precisely, let \mathbb{A}_K^\times denote the ideles of K , and for any \mathcal{O}_K -prime $\mathfrak{p} \mid c$, define

$$W_c(\mathfrak{p}) = 1 + \mathfrak{m}_{\mathcal{O}_{K_{\mathfrak{p}}}}^{c(\mathfrak{p})},$$

where $c(\mathfrak{p})$ denotes the largest power of \mathfrak{p} which contains $c\mathcal{O}_K$. Let

$$W_c = \prod_{\mathfrak{p} \mid c} W_c(\mathfrak{p}) \times \prod_{\mathfrak{p} \nmid c} \mathcal{O}_{K_{\mathfrak{p}}}^\times.$$

We can then define

$$\mathrm{Cl}(K, c) := \mathbb{A}_K^\times / K^\times \cdot W_c. \quad (2)$$

The fact that these two definitions are equivalent can be found, e.g. in Milne [11].

Let $K(c)$ denote the *ray class field of conductor c* of K , which is characterized as the unique abelian extension of K such that the Artin map provides an isomorphism between $\mathrm{Cl}(K, c)$ and $\mathrm{Gal}(K(c)/K)$. It is well-known that every finite abelian extension is contained in some ray class field. The *conductor* of a finite abelian extension L/K is defined to be the conductor of the smallest ray class field of K that L lies in (note that if $c \mid c'$, then $K(c) \subset K(c')$). Additionally, it is true that any prime \mathfrak{p} of \mathcal{O}_K that ramifies in L must divide c .

The importance of the conductor of a finite abelian extension is that it determines which primes are allowed to ramify. We next show that conductors of cubic cyclic extensions over a quadratic field are squarefree away from 3, and at most divisible by 9 if 3 is allowed to ramify.

Lemma 2.1. *Fix a integer $c \in \mathbb{Z}$ with prime factorization $c = 3^k \cdot \prod_{j=1}^n p_j^{k_j}$.*

- (a) *If $k = 0$, any cubic cyclic extension of a quadratic field K that is unramified away from primes dividing c is contained in $K(\prod_{j=1}^n p_j)$.*
- (b) *If $k > 0$, any cubic cyclic extension of a quadratic field K that is unramified away from the primes dividing c is contained in $K(9 \cdot \prod_{j=1}^n p_j)$.*

Proof. Any cubic cyclic extension L of a quadratic field K corresponds to an index-3 subgroup $H \subset \mathbb{A}_K^\times$, and let the conductor of L over K be f . Adelicly, f is the conductor if and only if f is the smallest integer such that $H \supset (1 + f\hat{\mathcal{O}}_K) \cap \hat{\mathcal{O}}_K^\times$. Since H is index 3, the cubes are contained in H , i.e.,

$$\left(\hat{\mathcal{O}}_K^\times\right)^3 \subseteq H.$$

Recall that $\widehat{\mathcal{O}}_K^\times = \prod_{\mathfrak{p} \subset \mathcal{O}_K} \mathcal{O}_{K_{\mathfrak{p}}}^\times$. If $\mathfrak{p} \nmid 3$, then $(\mathcal{O}_{K_{\mathfrak{p}}}^\times)^3$ contains the profinite p -group $1 + p\mathcal{O}_{K_{\mathfrak{p}}}$, and additionally, the map is surjective. This implies that the conductor f is squarefree away from 3, and we deduce part (a). If $\mathfrak{p} \mid 3$, then $(\mathcal{O}_{K_{\mathfrak{p}}}^\times)^3$ contains $1 + 9\mathcal{O}_{K_{\mathfrak{p}}} = (1 + 3\mathcal{O}_{K_{\mathfrak{p}}})^3$, thus $9 \mid f$, but $27 \nmid f$. We can then combine this with part (a) to deduce part (b). \square

Lemma 2.1 implies that the minimal conductors of cubic extensions of quadratic fields are either integers c such that c is squarefree or $\frac{c}{9} \in \mathbb{Z}$ is squarefree. We next study the relationship between conductors of certain cubic extensions and discriminants of their cubic subfields.

Lemma 2.2. (a) *Let K be a quadratic field. If L is a non-Galois cubic field such that the compositum LK is Galois over \mathbb{Q} , then $\text{Disc}(L) = \text{Disc}(K)f^2$ where f is the conductor of LK over K .*

(b) *If L is a Galois cubic field and $\text{Disc}(L) = f_0^2$, then $f_0 = 3^e \cdot p_1 \cdots p_m$ where $e = 0$ or 2 and each p_i denotes a distinct prime satisfying $p_i \equiv 1 \pmod{3}$ for all i . Additionally, $L \subset \mathbb{Q}(f_0)$.*

Proof. Part (a) follows from [10] or Thm. 9.2.6 of [4]. Part (b) follows from class field theory. \square

Next, we explicitly determine cubic fields that lie inside a cubic cyclic extension of a quadratic field K in order to count the size of the 3-torsion subgroup of the ray class group of K .

2.2 Index-3 subgroups of ray class groups of quadratic fields

For the remainder of the section, fix a conductor c as described by Lemma 2.1, i.e. set $c = c'$ or $c = 9c'$ where c' is a squarefree integer. We describe the relationship between index-3 subgroups of $\text{Cl}(K_2, c)$ for a quadratic field K_2 and certain pairs of cubic field. To do so, we first introduce some notation. Call an integer $d \in \mathbb{Z}$ *fundamental* if it is the discriminant of some quadratic field.

Definition 2.3. *We say that a pair of fields (K^+, K^-) is **c-valid** if:*

- $K^+ = \mathbb{Q}$ or a Galois cubic field with $\text{disc}(K^+) \mid c^2$, and
- $K^- = \mathbb{Q}$ or a non-Galois cubic field with $\text{disc}(K^-) = df^2$ where $d \in \mathbb{Z}$ is a fundamental and $f \mid c$.

*Two c-valid pairs (K^+, K^-) and (M^+, M^-) are **isomorphic** if both $K^+ \cong M^+$ and $K^- \cong M^-$.*

We give some examples of c -valid pairs below.

Examples.

- (a) The pair (\mathbb{Q}, \mathbb{Q}) is the *trivial* c -valid pair for any c .
- (b) For any cyclic cubic field K_3 satisfying $\text{Disc}(K_3) \mid c^2$, (K_3, \mathbb{Q}) is a c -valid pair. We will sometimes refer to K_3 as a *(cyclic) c-valid cubic field*.
- (c) For any non-cyclic cubic field K_3 whose discriminant can be written as df^2 where $f \mid c$ and d is a fundamental, (\mathbb{Q}, K_3) is a c -valid pair. We will refer to K_3 as a *(non-cyclic) c-valid cubic field*.
- (d) Let $c = 7$ and take $K^+ = \mathbb{Q}(\zeta_7 + \zeta_7^{-1})$, where ζ_7 denotes a 7th root of unity. Additionally, if θ denotes a root of $f(x) = x^3 - x^2 + 5x + 1$, then $K^- = \mathbb{Q}(\theta)$ totally ramifies at 7. Since $\text{Disc}(K^+) = 49$ and $\text{Disc}(K^-) = -3 \cdot 2^2 \cdot 7^2$, it follows that (K^+, K^-) is a 7-valid pair.

We now state the main result of this section, which describes the correspondence between c -valid pairs and index-3 subgroups of $\text{Cl}(K_2, c)$. This will allow us to later determine the size of $\text{Cl}_3(K_2, c)$ in terms of c -valid cubic fields.

Theorem 2.4. *Let $c = c'$ or $c = 9c'$ for any squarefree integer c' . There is a natural bijection between pairs (K_2, G) consisting of a quadratic field K_2 along with an index-3 subgroup G of $\text{Cl}(K_2, c)$ and isomorphism classes of non-trivial c -valid pairs.*

When $c = 1$ and $\text{Cl}_3(K_2) = \text{Cl}_3(K_2, 1)$, Theorem 2.4 is simply the bijection used in [7] between nowhere totally ramified cubic fields and index-3 subgroups of the class groups of quadratic fields (see also [2]). We prove this generalization by studying prime ramification in (cubic) subfields contained within the Galois closure of an arbitrary cubic cyclic extension K_6 over K_2 . These cubic subfields turn out to be c -valid iff K_6 is unramified away from c .

The goal for the remainder of this section is to prove Theorem 2.4. We first discuss the Galois theory of a cubic cyclic extension of a quadratic number field.

2.3 Cubic cyclic extensions of quadratic fields

In order to prove Theorem 2.4, we first show that for a fixed quadratic field, any cubic cyclic extension of conductor c is determined by a (unique up to isomorphism) non-trivial c -valid pair. To find a candidate for this c -valid pair, we look within the Galois closure (over \mathbb{Q}) of such sextic fields. For any number field K , let \widetilde{K} denote its Galois closure over \mathbb{Q} .

Fix a quadratic extension K_2/\mathbb{Q} . If K_6/K_2 is a cyclic cubic extension, then the Galois group $\text{Gal}(\widetilde{K}_6/\mathbb{Q})$ is equal to S_3 , C_6 , or $S_3 \times C_3$, which are the transitive subgroups with order at least 6 in the wreath product

$$\text{Gal}(K_6/K_2) \wr \text{Gal}(K_2/\mathbb{Q}) \cong (C_3 \times C_3) \rtimes C_2 \cong S_3 \times C_3. \quad (3)$$

Note that in the first two cases, K_6 is already Galois. We have the following field diagram when $K_6 \neq \widetilde{K}_6$.

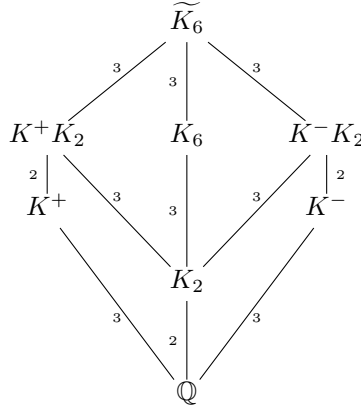


Figure 1: Some subfields of \widetilde{K}_6 when $\text{Gal}(K_6/\mathbb{Q}) \cong S_3 \times C_3$

When K_6 is not Galois, denote the subfield of \widetilde{K}_6 fixed by $C_2 \times C_3 \subset S_3 \times C_3$ as K^- , and the subfield fixed by S_3 as K^+ . We then have that $\text{Gal}(K^+/\mathbb{Q}) = C_3$, and $\text{Gal}(\widetilde{K}^-/\mathbb{Q}) = S_3$ with $\widetilde{K}^- = K^-K_2$. It follows that $\widetilde{K}_6 = \widetilde{K^+K^-}$. We have thus proven the following lemma stating that K^+ and K^- determine K_6 and vice versa.

Lemma 2.5. *Let K_6 denote a cubic cyclic extension over a quadratic field K_2 . If K_6 is not Galois over \mathbb{Q} , then*

$$\text{Gal}(\widetilde{K}_6/\mathbb{Q}) \cong S_3 \times C_3.$$

Additionally, there exists a unique pair of (isomorphism classes of) cubic subfields K^+ and K^- , where K^+ is cyclic and K^- is not Galois such that the Galois closure of K^+K^- is equal to \widetilde{K}_6 . In particular, we can

explicitly write

$$\widetilde{K}_6 = K^+ K_2 K^-.$$

This lemma implies that any degree-18 field with Galois group over \mathbb{Q} equal to $S_3 \times C_3$ is either determined by a degree 6 non-Galois subfield that is cyclic over a quadratic subextension or equivalently, by the fixed field of $C_2 \times C_3$ and the fixed field of S_3 . We will use this to prove that the pair of fields denoted in Figure 1 by K^+ and K^- make a c -valid pair whenever K_6 is unramified away from c .

2.4 Ramification in fields with Galois group $S_3 \times C_3$

We next show that a pair of cubic fields associated to a cubic cyclic extension of conductor c over a fixed quadratic field by Lemma 2.5 is indeed c -valid. We do so by understanding how ramification in the sextic field determines ramification in the pair and vice versa. We begin by reviewing properties of the discriminants of subfields within a Galois sextic field.

Lemma 2.6. *Fix a quadratic field K_2 . Any c -valid cubic field K such that $K_2 K$ is Galois over \mathbb{Q} satisfies:*

- (a) $\text{Disc}(K)^2 \mid \text{Disc}(K_2 K)$;
- (b) $\text{Nm}_K(\text{Disc}(K_2 K/K)) \mid \text{Disc}(K_2)^3$;
- (c) $\text{Disc}(K_2 K) \mid c^4 \cdot \text{Disc}(K_2)^3$.

Proof. There are two different ways we can calculate the discriminant of $K_2 K$ using the field towers $K_2 K/K_2/\mathbb{Q}$ and $K_2 K/K/\mathbb{Q}$:

$$\text{Nm}_{K_2}(\text{Disc}(K_2 K/K_2)) \cdot \text{Disc}(K_2)^3 = \text{Disc}(K_2 K) = \text{Nm}_K(\text{Disc}(K_2 K/K)) \cdot \text{Disc}(K)^2. \quad (4)$$

- (a) By the second equality in (4), we conclude that $\text{Disc}(K)^2 \mid \text{Disc}(K_2 K)$.
- (b) Let $[\beta_1, \beta_2]$ denote an integral basis of \mathcal{O}_{K_2} , and define the 2×2 matrix $M = [\sigma_i(\beta_j)]_{i,j}$ where σ_i run through elements of $\text{Gal}(K_2 K/K)$. First, we know that $\det(M)^2 = \text{Disc}(K_2)$, and second, we have that $[\beta_1, \beta_2]$ is a K -basis for $K_2 K$. This implies that $\det(M)^2 \in \text{Disc}(K_2 K/K)$, and hence

$$\text{Disc}(K_2 K)/K \mid \text{Disc}(K_2)$$

as \mathcal{O}_K -ideals. Combined with (4) and taking norms, we then conclude that

$$\text{Nm}_K(\text{Disc}(K_2 K/K)) \mid \text{Disc}(K_2)^3.$$

- (c) The extension $K_2 K/K_2$ is abelian, and thus it has a minimal modulus whose finite part is an integral K_2 -ideal denoted \mathfrak{m} . It is related to the relative discriminant by $\text{Disc}(K_2 K/K_2) = \mathfrak{m}^2$. Note that \mathfrak{m} is $\text{Gal}(K_2/\mathbb{Q})$ -invariant, and so $\text{Nm}_{K_2}(\mathfrak{m}^2) = \mathfrak{m}^4 \subseteq \mathbb{Z}$. If $\text{Disc}(K) = df^2$ where d is fundamental, then $\mathfrak{m} = f$ by Lemma 2.2(a), and so $\mathfrak{m} \mid c$. If $\text{Disc}(K) = f_0^2$ where $f_0 \mid c$, then by part (a) and (4), we have that $\text{Nm}_{K_2}(\mathfrak{m}^2) \mid f_0^4$, and so $\mathfrak{m} \mid c$.

From (4), we have that

$$\text{Disc}(K_2 K) = \text{Nm}_{K_2}(\mathfrak{m}^2) \cdot \text{Disc}(K_2)^3.$$

Since $\mathfrak{m} \mid c$, we obtain $\text{Disc}(K_2 K) \mid \text{Nm}_{K_2}(c^2) \cdot \text{Disc}(K_2)^3$, which implies that

$$\text{Disc}(K_2 K) \mid c^4 \cdot \text{Disc}(K_2)^3.$$

□

Proposition 2.7. *Fix a quadratic field K_2 . Any cubic cyclic extension K_6 over K_2 of conductor c that is not Galois over \mathbb{Q} has a c -valid pair (K^+, K^-) contained within the Galois closure $\widetilde{K_6}$ satisfying $\widetilde{K^+ K^-} = \widetilde{K_6}$. It is unique up to isomorphism.*

Proof. Given a cubic cyclic extension K_6 over K_2 of conductor c , the candidate c -valid pair (K^+, K^-) associated to K_6 comes from Lemma 2.5 and Figure 1. It remains to show that $\text{disc}(K^+) \mid c^2$ and $\frac{\text{disc}(K^-)}{\text{disc}(K_2)} \mid c^2$. To do so, we study the relationship between total ramification in K^+ or K^- and ramification in K_6/K_2 .

We begin with K^- . By Lemma 2.2(a), the conductor of K^-K_2/K_2 is equal to f where $\text{Disc}(K^-) = \text{Disc}(K_2)f^2$. Combined with Lemma 2.6, we obtain $\text{Disc}(K_2)^2 f^4 \mid c^4 \text{Disc}(K_2)^3$, and so

$$f^4 \mid c^4 \cdot \text{Disc}(K_2).$$

Recall that $\text{Disc}(K_2)$ is squarefree away from 2, and $2^4 \nmid \text{Disc}(K_2)$ for any quadratic field, so we conclude that $f \mid c$.

We now turn to K^+ . Lemma 2.6 implies that

$$\text{Disc}(K^+)^2 \mid c^4 \text{Disc}(K_2)^3.$$

In this case, $\text{Disc}(K^+) = f_0^2$ for some integer f_0 , and so we obtain

$$f_0^4 \mid c^4 \cdot \text{Disc}(K_2)^3.$$

If $\text{Disc}(K_2)$ is odd, then it is squarefree by Lemma 2.2(b). This implies $f_0 \mid c$.

If $\text{Disc}(K_2)$ is even, but $2 \nmid f_0$, then we also conclude $f_0 \mid c$. If $2 \mid f_0$, assume for the sake of contradiction that $2 \nmid c$. Then $\widetilde{K_6}$ is unramified over K_2 at the primes above 2. This implies that the ramification degree of $\widetilde{K_6}/\mathbb{Q}$ is at most 2 for $p = 2$, and thus the ramification degree is at most 2 in K^+ . Since K^+ is cyclic of degree 3, 2 is thus unramified in K^+ , which contradicts $2 \mid f_0$ since $\text{Disc}(K^+) = f_0^2$. Thus, $2 \mid c$, and since $4 \nmid f_0$ by Lemma 2.2(b), we obtain $f_0 \mid c$. \square

2.5 Proof of Theorem 2.4

We finally return to Theorem 2.4 and give a proof. We begin by giving an explicit description of the map. Let c' be a squarefree integer, and take $c = c'$ or $9c'$. If (K_2, G) is a quadratic field along with an index-3 subgroup G of $\text{Cl}(K_2, c)$, then let K_6 denote the fixed field in $K_2(c)$ for the subgroup G so that $\text{Gal}(K_6/K_2) = \text{Cl}(K_2, c)/G$. We then have:

- If K_6 is Galois over \mathbb{Q} and $\text{Gal}(K_6/\mathbb{Q}) = C_6$, then (K_2, G) corresponds to (K_3, \mathbb{Q}) , where K_3 is the cubic subfield of K_6 ;
- If K_6 is Galois over \mathbb{Q} and $\text{Gal}(K_6/\mathbb{Q}) = S_3$, then (K_2, G) corresponds to (\mathbb{Q}, K_3) , where K_3 is the cubic subfield of K_6 ;
- If K_6 is not Galois over \mathbb{Q} , then (K_2, G) corresponds to (K^+, K^-) as constructed in Lemma 2.5.

If K_6 is Galois, then its cubic subfield K_3 can only totally ramify at primes dividing c . Indeed, this is clear when $\text{Gal}(K_6/\mathbb{Q}) = C_6$. When $\text{Gal}(K_6/\mathbb{Q}) = S_3$, a prime p ramifies in the extension K_6/K_2 if and only if $p^2 \mid \frac{\text{Disc}(K_3)}{\text{Disc}(K_2)}$. Thus, K_3 is a c -valid cubic field in both cases. When K_6 is not Galois, Proposition 2.7 proves the forward direction.

To prove the other direction, we begin with a c -valid pair (K^+, K^-) and construct a cubic cyclic extension over K_2 of conductor dividing c . The compositum $K^+K_2K^-$ is Galois over \mathbb{Q} of degree 6 or 18. If it is degree 6, then (K^+, K^-) is in fact a c -valid *cubic field*, and we simply take K_6 to be the compositum $K^+K_2K^-$. In

this case, it remains to show that K_6 has conductor dividing c over K_2 , i.e. $K_6 \subset K_2(c)$. If $K^+K^- = K^+$, then by assumption $\text{Disc}(K^+) = f_0^2$ where $f_0 \mid c$; thus, $K^+ \subset \mathbb{Q}(f_0) \subset \mathbb{Q}(c)$, so $K_6 = K^+K_2 \subset K_2(c)$. If $K^+K^- = K^-$, then $\text{Disc}(K^-) = \text{Disc}(K_2)f^2$ where $f \mid c$, and so $K^-K_2 = \widetilde{K^-}$ is a cubic extension of K_2 contained in $K_2(f) \subset K_2(c)$ by Lemma 2.2(a).

If $K^+K_2K^-$ is degree 18, it has Galois group equal to $S_3 \times C_3$, and we define K_6 to be the fixed field of any non-normal $C_3 \subset S_3 \times C_3$. It remains to show that K_6 has conductor dividing c as an extension over K_2 . We do so by proving that K_2K^- and K_2K^+ have conductor dividing c over K_2 . Lemma 2.2(a) implies that K_2K^- has conductor f where $\text{Disc}(K^-) = \text{Disc}(K_2) \cdot f^2$, and $f \mid c$. Additionally, if $\text{Disc}(K^+) = f_0^2$, suppose p is a prime such that $p \nmid f_0$. Then p cannot ramify in K^+ , which implies that K^+K_2/K_2 is unramified above p . By Lemma 2.1, we conclude that K_2K^+/K_2 has conductor dividing

$$\begin{cases} \prod_{3 \nmid p \mid f_0} p & \text{if } 3 \nmid f_0, \text{ or} \\ 9 \cdot \prod_{3 \nmid p \mid f_0} p & \text{if } 3 \mid f_0. \end{cases}$$

If $3 \nmid f_0$, then f_0 is squarefree by Lemma 2.2(b), and so the conductor of K_2K^+ over K_2 divides c since $f_0 \mid c$. If $3 \mid f_0$, note that $9 \parallel f_0$; thus, we altogether obtain that K_2K^+ and K_2K^- both have conductor dividing c . Since $\widetilde{K_6} = K^+K_2K^-$, it must have conductor dividing c as an extension over K_2 , and so K_6 as a subextension must also have conductor dividing c .

It is easy to check that if the c -valid pair (K^+, K^-) is not isomorphic to (K_0^+, K_0^-) , then they correspond to non-isomorphic cubic cyclic extensions of K_2 of conductor c , and thus, they correspond to distinct index 3-subgroups of $\text{Cl}(K_2, c)$. \square

Using the fact that the number of order-3 subgroups is equal to the number of index-3 subgroups in a finite abelian group, we directly relate the number of non-trivial c -valid pairs to the number of 3-torsion elements in ray class groups of conductor c .

Corollary 2.8. *If c' is a squarefree integer, and $c = c'$ or $c = 9c'$, then*

$$\#\text{Cl}_3(K_2, c) = 2 \cdot \# \left\{ \begin{array}{l} \text{non-trivial } c\text{-valid pairs of fields } (K^+, K^-) \\ \text{s.t. } K^+ = \mathbb{Q} \text{ or has quadratic resolvent } K_2 \end{array} \right\} + 1.$$

Recall that the *quadratic resolvent* of a non-Galois cubic field K_3 is the quadratic subfield of the Galois closure $\widetilde{K_3}$. If K_3 has quadratic resolvent K_2 , then $\text{Disc}(K_2) \mid \text{Disc}(K_3)$ and $\frac{\text{Disc}(K_3)}{\text{Disc}(K_2)}$ is equal to the square of the conductor of $\text{Gal}(\widetilde{K_3}/K_2)$ by Lemma 2.2(b).

2.6 The action of $\text{Gal}(K_2/\mathbb{Q})$ on $\text{Cl}_3(K_2, c)$

We next consider the action of $\text{Gal}(K_2/\mathbb{Q})$ on $\text{Cl}_3(K_2, c)$. The number of c -valid *cubic fields* on the right-hand side of the equality in Corollary 2.8 is related to the sizes of eigenspaces for the action of $\text{Gal}(K_2/\mathbb{Q})$. Note that $\text{Cl}_3(K_2, c)$ is a $\text{Gal}(K_2/\mathbb{Q})$ -module of odd order, and thus we have two well-defined submodules of $\text{Cl}_3(K_2, c)$:

$$\begin{aligned} \text{Cl}_3^+(K_2, c) &:= \{[I] \in \text{Cl}_3(K_2, c) : \sigma(I) = I\}, \text{ and} \\ \text{Cl}_3^-(K_2, c) &:= \{[I] \in \text{Cl}_3(K_2, c) : \sigma(I) = J \text{ where } [I]^{-1} = [J]\}. \end{aligned}$$

We then have $\text{Cl}_3(K_2, c) = \text{Cl}_3^+(K_2, c) \oplus \text{Cl}_3^-(K_2, c)$, and thus

$$\#\text{Cl}_3(K_2, c) = \#\text{Cl}_3^+(K_2, c) + \#\text{Cl}_3^-(K_2, c). \quad (5)$$

Proposition 2.9. *Fix a quadratic field K_2 , and for any squarefree integer c' , let $c = c'$ or $c = 9c'$. Then:*

- (a) $\#\text{Cl}_3^+(K_2, c) = 2 \cdot \#\{\text{Cyclic } c\text{-valid cubic fields } K^+\} + 1;$
- (b) $\#\text{Cl}_3^-(K_2, c) = 2 \cdot \#\{\text{Non-cyclic } c\text{-valid cubic fields } K^- \text{ with quadratic resolvent } K_2\} + 1.$

Proof. The second part follows from Lemma 1.10 of [12] and Proposition 35 of [3]. To prove the first part, consider some cyclic cubic field K^+ unramified away from c . By class field theory and the proof of Proposition 2.4, K^+K_2/K_2 corresponds to a index-3 subgroup H of $\text{Cl}(K_2, c)[3]$, the 3-Sylow subgroup of $\text{Cl}(K_2, c)$. Since K^+K_2 is Galois over \mathbb{Q} , H has an action of $\text{Gal}(K_2/\mathbb{Q})$. Artin reciprocity implies that

$$\sigma(K^+K_2) = K^+K_2 \quad \Rightarrow \quad \sigma(H) = H.$$

Thus, we write $H = H^+ \oplus H^-$, where $H^\pm := \{[I] \in H : \sigma([I]) = [I]^\pm\}$. Let $\text{Cl}(K_2, c)[3]^\pm$ be defined analogously. Since H is index 3, it is clear that $H^+ = \text{Cl}(K_2, c)[3]^+$ or $H^- = \text{Cl}(K_2, c)[3]^+$.

We now show that $H^- = \text{Cl}(K_2, c)[3]^+$, so that $\text{Cl}(K_2, c)[3]/H \cong \text{Cl}(K_2, c)[3]^+/H^+$. For any lift $\tilde{\sigma}$ of σ to $\text{Gal}(K^+K_2/K_2)$, Artin reciprocity implies the action of conjugation on $\text{Gal}(K^+K_2/K_2)$ by $\tilde{\sigma}$ corresponds to acting by σ on $\text{Cl}(K_2, c)[3]/H$. Since $\text{Gal}(K^+K_2/K_2)$ is isomorphic to C_6 , σ acts trivially on $\text{Cl}(K_2, c)[3]/H$ so $H^- = \text{Cl}(K_2, c)[3]^+$. We then have that the number of index-3 subgroups of $\text{Cl}(K_2, c)[3]^+$ is the same as the number of order-3 subgroups, which are generated by nontrivial elements of $\text{Cl}_3(K_2, c)^+$. Since powers of an element generate the same subgroup we then deduce the first part. \square

We additionally remark that $\text{Cl}_3^+(K_2, c) = \text{Cl}_3(\mathbb{Q}, c)$. This implies that $\text{Cl}_3^+(K_2, c)$ is independent of K_2 . This is a crucial fact that greatly simplifies the computation for the average size of ray class groups of conductor c when K_2 is allowed to vary. We next compute asymptotics for both $\text{Cl}_3^\pm(K_2, c)$ by counting the relevant c -valid cubic fields as given in Proposition 2.9.

3 Counting c -valid cubic fields of bounded discriminant

The results of the previous section allow us to determine the number of 3-torsion elements in ray class groups of quadratic fields simply in terms of c -valid cubic fields instead of c -valid pairs. We first compute the size of $\text{Cl}_3^+(K_2, c)$ for any quadratic field K_2 by enumerating the number of cyclic c -valid cubic fields. In order to obtain asymptotics for the size of $\text{Cl}_3^-(K_2, c)$, we then employ the results of [2] building on those of [7] for computing the number of cubic fields with bounded discriminant that satisfy certain ramification restrictions.

3.1 The size of the 3-torsion subgroup in ray class groups of \mathbb{Q}

As before, let K_2 be a quadratic field. In this section, we prove that the number of $\text{Gal}(K_2/\mathbb{Q})$ -stable elements in the 3-torsion subgroups of the ray class group of conductor c depends only on the number of distinct primes congruent to 1 mod 3 that divide c . More precisely,

Proposition 3.1. *Let K_2 be a quadratic field, and let c be a positive integer. Let the number of distinct prime factors $p_i \mid c$ such that $p_i \not\equiv 2 \pmod{3}$ is m . Then*

$$\#\text{Cl}_3^+(K_2, c) = \begin{cases} 3^m & \text{if } c \not\equiv 3, 6 \pmod{9}, \text{ and} \\ 3^{m-1} & \text{if } c \equiv 3, 6 \pmod{9}, \end{cases}$$

independent of the quadratic field K_2 .

Proof. By Proposition 2.9(a), we enumerate elements of $\text{Cl}_3^+(K_2, c)$ by counting the normal cubic extensions of \mathbb{Q} with discriminant dividing c^2 . If K^+ is such a cyclic field of degree 3, by Lemma 2.2(b), the conductor of K^+ is equal to $c_0 = 3^e \cdot p_1 \cdots p_m$ where $e = 0$ or 2 , and p_i denotes distinct primes satisfying $p_i \equiv 1 \pmod{3}$ for all i . Furthermore, if $e = 0$, then there are 2^{m-1} cubic cyclic fields of conductor c_0 , and if $e = 2$, then there are 2^m cubic cyclic fields of conductor c_0 (see [6]). Thus, we are counting cubic fields with discriminant c_0^2 where $c_0 \mid c$ and no prime factor of c_0 is congruent to $2 \pmod{3}$.

If $3 \nmid c$, let $c = p_1^{k_1} \cdots p_m^{k_m} \cdot q_{m+1}^{k_{m+1}} \cdots q_n^{k_n}$ where each p_i is a distinct prime congruent to $1 \pmod{3}$, and q_j are distinct primes congruent to $2 \pmod{3}$. There is one abelian extension with conductor p_i for each p_i , two abelian extensions with conductor $p_i p_k$ for each pair of primes, four abelian extensions with conductor $p_i p_k p_l$ for each triplet of primes, and so on. Thus, we obtain

$$\#\text{Cl}_3^+(K_2, c) = 1 + 2 \cdot \left(\sum_{j=1}^m \binom{m}{j} 2^{m-1} \right) = 3^m.$$

Similarly, if $c = 3^k \cdot p_1^{k_1} \cdots p_m^{k_m} \cdot q_{m+1}^{k_{m+1}} \cdots q_n^{k_n}$ where $k \geq 2$, we deduce

$$\#\text{Cl}_3^+(K_2, c) = 1 + 2 \cdot \left(\sum_{j=1}^{m+1} \binom{m+1}{j} 2^m \right) = 3^{m+1}.$$

If $k = 1$, we obtain,

$$\#\text{Cl}_3^+(K_2, c) = 1 + 2 \cdot \left(\sum_{j=1}^m \binom{m+1}{j} 2^{m-1} \right) = 3^m.$$

□

3.2 The number of cubic fields with prescribed total ramification

We want to next determine the asymptotics for the number of cubic fields that are totally ramified at a certain fixed set of primes. Let \mathcal{K} denote the set of isomorphism classes of cubic fields, and for any subset $\mathcal{K}' \subseteq \mathcal{K}$, define for $i = 0$ or 1 :

$$N_3^{(i)}(\mathcal{K}', X) := \#\{K_3 \in \mathcal{K}' \mid 0 < (-1)^i \text{Disc}(K_3) < X\}.$$

Theorem 3.2. *Let S denote a set of primes, and let $n_i = |\text{Aut}(\mathbb{R}^{3-2i} \oplus \mathbb{C}^i)|$ for $i = 0$ or 1 .*

- (a) *Let \mathcal{K}_S denote the set of isomorphism classes of cubic fields that are totally ramified exactly at the primes $p \in S$.*

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S, X)}{X} = \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}.$$

- (b) *If $3 \in S$, let $\mathcal{K}_S^{(3)}$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and have discriminant that is not divisible by 81 .*

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(3)}, X)}{X} = \frac{2}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}.$$

- (c) *If $3 \in S$, let $\mathcal{K}_S^{(9)}$ denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and have discriminant divisible by 81 .*

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(9)}, X)}{X} = \frac{1}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}.$$

- (d) Let S_0 be a set of primes containing S . Let \mathcal{K}_{S,S_0} denote the set of isomorphism classes of cubic fields that are totally ramified exactly at $p \in S$ and unramified at $p \in S_0 \setminus S$.

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_{S,S_0}, X)}{X} = \frac{3}{n_i \pi^2} \cdot \prod_{p \in S_0 \setminus S} \frac{p}{p+1} \cdot \prod_{p \in S} \frac{1}{p(p+1)}.$$

- (e) Let S_0 be a set of primes containing S . If $3 \in S$, let $\mathcal{K}_{S,S_0}^{(9)}$ denote the set of isomorphism classes of cubic fields K_3 that are totally ramified exactly at $p \in S$, unramified at $p \in S_0 \setminus S$, and $\text{disc}(K_3) \parallel 81$.

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_{S,S_0}^{(9)}, X)}{X} = \frac{1}{n_i \pi^2} \cdot \frac{2}{3} \cdot \prod_{p \in S_0 \setminus S} \frac{p}{p+1} \cdot \prod_{p \in S} \frac{1}{p(p+1)}.$$

Proof. For any prime p , let Σ_p^{tr} denote the set of all isomorphism classes of maximal cubic étale algebras over \mathbb{Z}_p that are totally ramified. Let Σ_p^{ur} denote the set of all isomorphism classes of maximal cubic étale algebras over \mathbb{Z}_p that are unramified. Additionally, let Σ_p^{ntr} denote the set of all isomorphism classes of maximal cubic étale algebras over \mathbb{Z}_p that are not totally ramified. When $p = 3$, let $\Sigma_3^{(3)}$ denote the subset of Σ_3^{tr} whose discriminant over \mathbb{Z}_p is not divisible by 81, and let $\Sigma_3^{(9)}$ denote the subset of Σ_3^{tr} whose discriminant is divisible by 81. Finally, let $\Sigma_3^{(81)}$ denote the subset of Σ_3^{tr} whose discriminant is equal to 81.

We will take Σ to be the set of isomorphism classes of rings of integers inside cubic fields contained in (a) \mathcal{K}_S , (b) $\mathcal{K}_S^{(3)}$, (c) $\mathcal{K}_S^{(9)}$, (d) \mathcal{K}_{S,S_0} , or (e) $\mathcal{K}_{S,S_0}^{(9)}$, respectively. In each case, Σ is determined by an *acceptable* collection of local specifications as defined in [2]. Indeed, we can equivalently define Σ to be the set of all isomorphism classes of maximal orders R in cubic fields for which $R \otimes \mathbb{Z}_p \in \Sigma_p$ for all p where:

$$\begin{aligned} \text{(a)} \quad \Sigma_p &= \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S; \end{cases} & \text{(d)} \quad \Sigma_p &= \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S_0, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S, \\ \Sigma_p^{\text{ur}} & \text{if } p \in S_0 \setminus S; \end{cases} \\ \text{(b)} \quad \Sigma_p &= \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \setminus \{3\}, \\ \Sigma_3^{(3)} & \text{if } p = 3; \end{cases} & \text{(e)} \quad \Sigma_p &= \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S_0, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \setminus \{3\}, \\ \Sigma_p^{\text{ur}} & \text{if } p \in S_0 \setminus S, \\ \Sigma_3^{(81)} & \text{if } p = 3. \end{cases} \\ \text{(c)} \quad \Sigma_p &= \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S \setminus \{3\}, \\ \Sigma_3^{(9)} & \text{if } p = 3; \end{cases} \end{aligned}$$

We can compute $N_3^{(i)}(\Sigma, X)$, the number of cubic rings $R \in \Sigma$ with $0 < (-1)^i \text{Disc}(R) < X$, using Theorem 7 in [2], which determines a mass formula for $N_3^{(i)}(\Sigma, X)$ whenever Σ is defined by an acceptable collection of local conditions. This implies that

$$\lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\Sigma, X)}{X} = \frac{1}{2n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{|\text{Aut}(R)|} \right), \quad (6)$$

where $\text{Disc}_p(R)$ denotes the discriminant of R over \mathbb{Z}_p as a power of p . One can compute (or combine

Lemmas 18, 19, and 32 in [2] to deduce)

$$\sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{|\text{Aut}(R)|} = \begin{cases} \frac{p+1}{p} & \text{if } \Sigma_p = \Sigma_p^{\text{ntr}}, \\ \frac{1}{p^2} & \text{if } \Sigma_p = \Sigma_p^{\text{tr}}, \\ 1 & \text{if } \Sigma_p = \Sigma_p^{\text{ur}}, \\ \frac{2}{27} & \text{if } \Sigma_3 = \Sigma_3^{(3)}, \\ \frac{1}{27} & \text{if } \Sigma_3 = \Sigma_3^{(9)}, \text{ and} \\ \frac{2}{81} & \text{if } \Sigma_3 = \Sigma_3^{(81)}. \end{cases}$$

Combining with (6), we can then compute:

$$\begin{aligned} \lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S, X)}{X} &= \frac{1}{2n_i} \cdot \prod_{p \notin S} \frac{p^2 - 1}{p^2} \cdot \prod_{p \in S} \frac{p - 1}{p^3} \\ &= \frac{1}{2n_i} \cdot \prod_p \left(1 - \frac{1}{p^2}\right) \cdot \prod_{p \in S} \frac{1}{p(p+1)} \\ &= \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}; \\ \lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(3)}, X)}{X} &= \frac{1}{2n_i} \cdot \frac{4}{81} \cdot \prod_{p \notin S} \frac{p^2 - 1}{p^2} \cdot \prod_{3 \neq p \in S} \frac{p - 1}{p^3} \\ &= \frac{1}{2n_i} \cdot \frac{2}{3} \cdot \prod_p \left(1 - \frac{1}{p^2}\right) \cdot \prod_{p \in S} \frac{1}{p(p+1)} \\ &= \frac{2}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}; \\ \lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_S^{(9)}, X)}{X} &= \frac{1}{2n_i} \cdot \frac{2}{81} \cdot \prod_{p \notin S} \frac{p^2 - 1}{p^2} \cdot \prod_{3 \neq p \in S} \frac{p - 1}{p^3} \\ &= \frac{1}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)}; \\ \lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_{S, S_0}, X)}{X} &= \frac{1}{2n_i} \cdot \prod_{p \notin S_0} \frac{p^2 - 1}{p^2} \cdot \prod_{p \in S_0 \setminus S} \frac{p - 1}{p} \cdot \prod_{p \in S} \frac{p - 1}{p^3} \\ &= \frac{1}{2n_i} \cdot \prod_p \left(1 - \frac{1}{p^2}\right) \cdot \prod_{p \in S_0 \setminus S} \frac{p}{p+1} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \\ &= \frac{3}{n_i \pi^2} \cdot \prod_{p \in S_0 \setminus S} \frac{p}{p+1} \cdot \prod_{p \in S} \frac{1}{p(p+1)}; \\ \lim_{X \rightarrow \infty} \frac{N_3^{(i)}(\mathcal{K}_{S, S_0}^{(9)}, X)}{X} &= \frac{1}{2n_i} \cdot \frac{4}{243} \cdot \prod_{p \notin S_0} \frac{p^2 - 1}{p^2} \cdot \prod_{p \in S_0 \setminus S} \frac{p - 1}{p} \cdot \prod_{3 \neq p \in S} \frac{p - 1}{p^3} \\ &= \frac{2}{3n_i \pi^2} \cdot \prod_{p \in S_0 \setminus S} \frac{p}{p+1} \cdot \prod_{p \in S} \frac{1}{p(p+1)}. \end{aligned}$$

□

3.3 Computing the mean size of $\text{Cl}_3^-(K_2, c)$ over quadratic fields K_2

In this section, we compute the average number of 3-torsion elements in the minus eigenspace of their ray class groups of fixed conductor c in families of quadratic fields ordered by discriminant.

Proposition 3.3. *Fix a positive integer c , and recall that $n_0 = 6$ and $n_1 = 2$. Then*

$$\begin{aligned}
\text{(a) If } 3 \nmid c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{0 < (-1)^i \text{Disc}(K_2) < X} 1} &= 1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1}\right); \\
\text{(b) If } 3 \parallel c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{0 < (-1)^i \text{Disc}(K_2) < X} 1} &= 1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1}\right); \\
\text{(c) If } 9 \mid c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{0 < (-1)^i \text{Disc}(K_2) < X} 1} &= 1 + \frac{30}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1}\right).
\end{aligned}$$

Proof. Let S_c denote the set of primes dividing c , and for shorthand, let

$$\text{Avg}^{(i)}(\text{Cl}_3^-(c)) := \lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_3^-(K_2, c)}{\sum_{0 < (-1)^i \text{Disc}(K_2) < X} 1}.$$

Recall that the density of fundamental discriminants is:

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1}{X} = \frac{3}{\pi^2}. \tag{7}$$

(a) Assume $3 \nmid c$. Proposition 2.9 combined with (7) implies that

$$\text{Avg}^{(i)}(\text{Cl}_3^-(c)) = 1 + 2 \cdot \frac{\pi^2}{3} \cdot \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c} N_3^{(i)}(\mathcal{K}_S, X \cdot \prod_{p \in S} p^2)}{X}.$$

By Theorem 3.2(a), we conclude that

$$\begin{aligned}
\text{Avg}^{(i)}(\text{Cl}_3^-(c)) &= 1 + 2 \cdot \frac{\pi^2}{3} \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \right) \\
&= 1 + \frac{2}{n_i} \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} \frac{p}{p+1} \right) \\
&= 1 + \frac{2}{n_i} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right).
\end{aligned}$$

This proves (a).

(b) Assume $3 \parallel c$. Proposition 2.9 and (7) imply that

$$\text{Avg}^{(i)}(\text{Cl}_3^-(c)) = 1 + \frac{2\pi^2}{3} \cdot \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_S, X \cdot \prod_{p \in S} p^2) + N_3^{(i)}(\mathcal{K}_{S \cup \{3\}}^{(3)}, X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X}.$$

By Theorem 3.2(a) and (b), we obtain

$$\text{Avg}^{(i)}(\text{Cl}_3^-(c)) = 1 + 2 \cdot \frac{\pi^2}{3} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} + \prod_{p \in S \cup \{3\}} p^2 \cdot \frac{2}{n_i \pi^2} \cdot \prod_{p \in S \cup \{3\}} \frac{1}{p(p+1)} \right).$$

Reducing the right-hand side, we obtain

$$\begin{aligned} \text{Avg}^{(i)}(\text{Cl}_3^-(c)) &= 1 + \frac{2}{n_i} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} \frac{p}{p+1} + \frac{2}{3} \cdot \prod_{p \in S \cup \{3\}} \frac{p}{p+1} \right) \\ &= 1 + \frac{2}{n_i} \cdot \left(\frac{1}{3} \cdot \prod_{p \in S_c \setminus \{3\}} \left(1 + \frac{p}{p+1} \right) + \frac{2}{3} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right) \right) \\ &= 1 + \frac{2}{n_i} \cdot \left(\frac{1}{3} \cdot \frac{4}{7} + \frac{2}{3} \right) \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right) \\ &= 1 + \frac{12}{7n_i} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right). \end{aligned}$$

(c) Assume $9 \mid c$. By Proposition 2.9 and (7), we have that

$$\begin{aligned} \text{Avg}^{(i)}(\text{Cl}_3^-(c)) &= 1 + \frac{2\pi^2}{3} \cdot \left(\lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_S, X \cdot \prod_{p \in S} p^2) + N_3^{(i)}(\mathcal{K}_S^{(3)}, X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X} \right. \\ &\quad \left. + \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_{S \cup \{3\}}^{(9)}, 9X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X} \right). \end{aligned}$$

Theorem 3.2(a) and (c) imply

$$\begin{aligned}
\text{Avg}^{(i)}(\text{Cl}_3^-(c)) &= 1 + 2 \cdot \frac{\pi^2}{3} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \right. \\
&\quad + \prod_{p \in S \cup \{3\}} p^2 \cdot \frac{2}{n_i \pi^2} \cdot \prod_{p \in S \cup \{3\}} \frac{1}{p(p+1)} \\
&\quad \left. + 9 \cdot \prod_{p \in S \cup \{3\}} p^2 \cdot \frac{1}{n_i \pi^2} \cdot \prod_{p \in S \cup \{3\}} \frac{1}{p(p+1)} \right) \\
&= 1 + \frac{2}{n_i} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} \frac{p}{p+1} + \frac{2}{3} \cdot \prod_{p \in S \cup \{3\}} \frac{p}{p+1} + 3 \cdot \prod_{p \in S \cup \{3\}} \frac{p}{p+1} \right) \\
&= 1 + \frac{2}{n_i} \cdot \left(-\frac{8}{3} \cdot \frac{4}{7} + \frac{11}{3} \right) \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right) \\
&= 1 + \frac{30}{7n_i} \cdot \prod_{p \in S_c} \left(1 + \frac{p}{p+1} \right).
\end{aligned}$$

□

3.4 Averaging over certain acceptable families of quadratic fields

Next, we vary over only those quadratic fields whose discriminants are coprime to the choice of fixed conductor. We first describe the asymptotic number of discriminants of quadratic fields that are relatively prime to a fixed integer.

Lemma 3.4. *Let c be a positive integer.*

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c) = 1 \\ 0 < (-1)^t \text{Disc}(K_2) < X}} 1}{X} = \frac{3}{\pi^2} \cdot \prod_{p|c} \frac{p}{p+1}.$$

Proof. By Proposition 2.2 and (4.2) in [1], we have that the number of real (resp. imaginary) quadratic fields that are unramified away from c is asymptotically

$$\frac{1}{2} \cdot \prod_{p \nmid c} \left(\frac{p-1}{p} \cdot \left(1 + \frac{1}{p} \right) \right) \cdot \prod_{p|c} \left(\frac{p-1}{p} \cdot 1 \right) \cdot X = \frac{3}{\pi^2} \cdot \left(\prod_{p|c} \frac{p}{p+1} \right) \cdot X.$$

□

Note that for non-Galois cubic fields K_3 that are totally ramified at 3, $\text{Disc}(K_3)$ is exactly divisible by 3^3 , 3^4 , or 3^5 , and in order for the quadratic resolvent K_2 of K_3 to have discriminant relatively prime to 3, $\text{Disc}(K_3) = \text{Disc}(K_2)f^2$ where either $3 \nmid f$ or $9 \parallel f$.

Proposition 3.5. *Fix a positive integer c , and let n denote the number of distinct primes dividing c . Recall that $n_0 = 6$ and $n_1 = 2$. Then*

$$(a) \text{ If } 3 \nmid c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = 1 + 2 \cdot \frac{2^n}{n_i};$$

$$(b) \text{ If } 3 \parallel c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = 1 + \frac{2^n}{n_i};$$

$$(c) \text{ If } 9 \mid c, \text{ then } \lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \text{Cl}_3^-(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = 1 + 3 \cdot \frac{2^n}{n_i}.$$

Proof. Let S_c denote the set of primes dividing c . For shorthand, let

$$\text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) := \lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_{3,}^-(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1}.$$

(a) Assume $3 \nmid c$. Proposition 2.9 combined with Lemma 3.4 implies that

$$\text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) = 1 + \frac{2\pi^2}{3} \cdot \left(\prod_{p \mid c} \frac{p+1}{p} \right) \cdot \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c} N_3^{(i)}(\mathcal{K}_{S, S_c}, X \cdot \prod_{p \in S} p^2)}{X}.$$

By Theorem 3.2(d), we conclude that

$$\begin{aligned} \text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) &= 1 + \frac{2\pi^2}{3} \cdot \prod_{p \mid c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \cdot \prod_{p \in S_c \setminus S} \frac{p}{p+1} \right) \\ &= 1 + \frac{2}{n_i} \cdot \prod_{p \mid c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} \frac{p}{p+1} \right) \\ &= 1 + \frac{2^{n+1}}{n_i}. \end{aligned}$$

(b) If $3 \parallel c$, note that any quadratic field K_2 that is unramified at 3 satisfies

$$\text{Cl}^-(K_2, c) = \text{Cl}^-(K_2, \frac{c}{3}).$$

Thus, Proposition 2.9 and Lemma 3.4 together imply that

$$\text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) = 1 + \frac{2\pi^2}{3} \cdot \left(\prod_{p|c} \frac{p+1}{p} \right) \cdot \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_{S, S_c}, X \cdot \prod_{p \in S} p^2)}{X}.$$

Using Theorem 3.2(d), we obtain

$$\begin{aligned} \text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) &= 1 + \frac{2\pi^2}{3} \cdot \prod_{p|c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \cdot \prod_{p \in S_c \setminus S} \frac{p}{p+1} \right) \\ &= 1 + \frac{2}{n_i} \cdot \prod_{p|c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S_c} \frac{p}{p+1} \right) \\ &= 1 + \frac{2^n}{n_i}. \end{aligned}$$

(c) If $9 \mid c$, by Proposition 2.9 and Lemma 3.4, we have

$$\begin{aligned} \text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) &= 1 + \frac{2\pi^2}{3} \cdot \left(\prod_{p|c} \frac{p+1}{p} \right) \cdot \left(\lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_{S, S_c}, X \cdot \prod_{p \in S} p^2)}{X} \right. \\ &\quad \left. + \lim_{X \rightarrow \infty} \frac{\sum_{S \subseteq S_c \setminus \{3\}} N_3^{(i)}(\mathcal{K}_{S \cup \{3\}, S_c}^{(9)}, 9X \cdot \prod_{p \in S \cup \{3\}} p^2)}{X} \right). \end{aligned}$$

Finally, by Theorem 3.2(d) and (e), we conclude that

$$\begin{aligned} \frac{3 \cdot (\text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) - 1)}{2\pi^2} &= \prod_{p|c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(\prod_{p \in S} p^2 \cdot \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \cdot \prod_{p \in S_c \setminus S} \frac{p}{p+1} \right) \\ &\quad + 9 \cdot \prod_{p \in S \cup \{3\}} p^2 \cdot \frac{2}{3n_i \pi^2} \cdot \prod_{p \in S \cup \{3\}} \frac{1}{p(p+1)} \cdot \prod_{p \in S_c \setminus (S \cup \{3\})} \frac{p}{p+1} \\ &= \frac{1}{n_i \pi^2} \cdot \prod_{p|c} \frac{p+1}{p} \cdot \sum_{S \subseteq S_c \setminus \{3\}} \left(3 \cdot \prod_{p \in S_c} \frac{p}{p+1} + 6 \cdot \prod_{p \in S_c \cup \{3\}} \frac{p}{p+1} \right) \\ &= \frac{1}{n_i \pi^2} \cdot 9 \cdot 2^{n-1}. \end{aligned}$$

This implies that

$$\text{Avg}_c^{(i)}(\text{Cl}_3^-(c)) = 1 + 3 \cdot \frac{2^n}{n_i}.$$

□

4 Proofs of Theorems 1-2 and Corollary 3

We put together the results of the previous sections in order to conclude Theorems 1 and 2, as well as Corollary 3. For Theorem 1, we will first allow K_2 to vary over all quadratic fields of bounded discriminant

and use the combination of Propositions 2.9, 3.1, and 3.3. Corollary 3 is then derived from Theorem 1. Afterwards, we only vary over the discriminants of quadratic fields that are coprime to the fixed conductor, and we combine Propositions 2.9, 3.1, and 3.5 in order to conclude Theorem 2.

4.1 Proof of Theorem 1

We combine Propositions 3.3 and 3.1 of the previous sections to prove the Theorem 1. Let c be an integer, and assume that there are m primes dividing c that are congruent to 1 mod 3. Define k to satisfy $3^k \parallel c$, and recall that $\text{Cl}_3^+(K_2, c)$ only depends on m and k . It is independent of the choice of quadratic field, so in particular, we have by Proposition 3.1,

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_3^+(K_2, c)}{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \#\text{Cl}_3^+(K_2, c) = \begin{cases} 3^m & \text{if } k \leq 1, \text{ and} \\ 3^{m+1} & \text{if } k \geq 2. \end{cases}$$

Thus,

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_3(K_2, x)}{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \#\text{Cl}_3^+(K_2, c) \cdot \lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_3^-(K_2, c)}{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} 1}.$$

We conclude by Proposition 3.3 and (5) that

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} \#\text{Cl}_3(K_2, c)}{\sum_{\substack{0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \begin{cases} 3^m \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) & \text{if } k = 0; \\ 3^m \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) & \text{if } k = 1; \\ 3^{m+1} \cdot \left(1 + \frac{30}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) & \text{if } k \geq 2. \end{cases} \quad (8)$$

4.2 Proof of Corollary 3

We next turn to the proof of Corollary 3. We use the Theorem 1 to find lower bounds for the proportion $P_1(i, c)$ of quadratic fields with i pairs of complex embeddings whose ray class groups of conductor c have trivial 3-torsion subgroup.

Assume $3 \nmid c$. If m denotes the distinct number of primes dividing c that are congruent to 1 mod 3, we have by (8),

$$3^m \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \geq 1 \cdot P_1(i, c) + 3 \cdot (1 - P_1(i, c)).$$

Hence,

$$\frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \leq P_1(i, c). \quad (9)$$

We now calculate for which c is $P_1(i, c) > 0$ using (9). Note that

$$\begin{aligned} 0 < \frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) &\Leftrightarrow 3 > 3^m \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \\ &\Leftrightarrow m = 0 \text{ and } n_i > \prod_{p|c} \left(1 + \frac{p}{p+1} \right). \end{aligned}$$

Thus, we conclude automatically that for any conductor c of the form $c = p$ where $p \equiv 2 \pmod{3}$, a positive proportion of real (resp. imaginary) quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c . Additionally, for any conductor c of the form $c = p_1 p_2$ where $p_i \equiv 2 \pmod{3}$, we see that a positive proportion of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c .

If $3 \parallel c$, and m denotes the distinct number of primes dividing c that are congruent to 1 mod 3, we have by (8),

$$3^m \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \geq 1 \cdot P_1(i, c) + 3 \cdot (1 - P_1(i, c)).$$

Hence,

$$\frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \leq P_1(i, c). \quad (10)$$

We then have by (10) that $P_1(i, c) > 0$ if

$$\begin{aligned} 0 < \frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) &\Leftrightarrow 3 > 3^m \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \\ &\Leftrightarrow m = 0 \text{ and } \frac{7n_i}{6} > \prod_{p|c} \left(1 + \frac{p}{p+1} \right). \\ &\Leftrightarrow m = 0 \text{ and } \frac{2n_i}{3} > \prod_{\substack{p|c \\ p \neq 3}} \left(1 + \frac{p}{p+1} \right) \end{aligned}$$

Thus, for real quadratic fields, if c is a product of 3 and at most two distinct primes which are congruent to 2 mod 3, a positive proportion of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor c . Additionally, a positive proportion of imaginary quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor 3.

Similarly, one can show that if $3 \nmid c$,

$$\frac{1}{2} \leq \frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{2}{n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \Leftrightarrow m = 0, \text{ and } \frac{n_i}{2} \geq \prod_{p|c} \left(1 + \frac{p}{p+1} \right).$$

Thus, at least 50% of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of prime conductor $c \equiv 2 \pmod{3}$. If $3 \parallel c$,

$$\frac{1}{2} \leq \frac{3}{2} - \frac{3^m}{2} \cdot \left(1 + \frac{12}{7n_i} \cdot \prod_{p|c} \left(1 + \frac{p}{p+1} \right) \right) \Leftrightarrow m = 0, \text{ and } \frac{n_i}{3} \geq \prod_{\substack{p|c \\ p \neq 3}} \left(1 + \frac{p}{p+1} \right).$$

Hence, at least 50% of real quadratic fields have trivial 3-torsion subgroup in their ray class groups of conductor $3p$ where $p \equiv 2 \pmod{3}$.

4.3 Proof of Theorem 2

In order to compute the average number of 3-torsion elements in ray class groups of fixed conductor of quadratic fields with discriminant that is both bounded and coprime to the choice of conductor, we use Proposition 3.1 and 3.5. Let c be an integer, and assume that there are n distinct primes dividing c , m of which are congruent to 1 mod 3. Define k to satisfy $3^k \parallel c$. By Proposition 3.1,

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \# \text{Cl}_3^+(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \# \text{Cl}_3^+(K_2, c) = \begin{cases} 3^m & \text{if } k \leq 1, \text{ and} \\ 3^{m+1} & \text{if } k \geq 2. \end{cases}$$

Thus by (5),

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \# \text{Cl}_3(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \# \text{Cl}_3^+(K_2, c) \cdot \lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \# \text{Cl}_3^-(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1}.$$

Combining with Proposition 3.5, we obtain

$$\lim_{X \rightarrow \infty} \frac{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} \# \text{Cl}_3(K_2, c)}{\sum_{\substack{(\text{Disc}(K_2), c)=1 \\ 0 < (-1)^i \text{Disc}(K_2) < X}} 1} = \begin{cases} 3^m \cdot \left(1 + \frac{2^{n+1}}{n_i}\right) & \text{if } k = 0, \\ 3^m \cdot \left(1 + \frac{2^n}{n_i}\right) & \text{if } k = 1, \text{ and} \\ 3^{m+1} \cdot \left(1 + 3 \cdot \frac{2^n}{n_i}\right) & \text{if } k \geq 2. \end{cases} \quad (11)$$

5 Second main term and the proof of Theorem 4

To compute the second main term for the mean number of 3-torsion elements in ray class groups of quadratic fields of bounded discriminant, we use a refinement of Theorem 3.2. For any set of primes S not containing 3, recall that \mathcal{K}_S denotes the set of isomorphism classes of cubic fields that are totally ramified *exactly* at the primes $p \in S$. We first introduce some notation from [2]. For a free \mathbb{Z}_p -module M , define $M^{\text{Prim}} \subset M$ by $M^{\text{Prim}} := M \setminus \{pM\}$. Also, for any element x in a cubic order, let $i(x) := [R : \mathbb{Z}_p[x]]$. As in the proof of Theorem 3.2, let Σ^S denote the set of all isomorphism classes of rings of integers of cubic fields in \mathcal{K}_S . Then, both Σ^S is *strongly acceptable* as defined in [2]. Thus, if $N_3^{(i)}(\Sigma^S, X)$ denotes the number of cubic orders $R \in \Sigma^S$ satisfying $0 < (-1)^i \text{Disc}(R) < X$, Theorem 7 of [2] determines the asymptotic count with two main terms:

$$\begin{aligned} N_3^{(i)}(\Sigma^S; X) &= \frac{1}{2n_i} \cdot \prod_p \left(\frac{p-1}{p} \cdot \sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{\# \text{Aut}(R)} \right) \cdot X \\ &\quad + \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \left((1 - p^{-1/3}) \cdot \sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{\# \text{Aut}(R)} \int_{(R/\mathbb{Z}_p)^{\text{Prim}}} i(x)^{2/3} dx \right) \cdot X^{5/6} \\ &\quad + O_\epsilon(X^{5/6-1/48+\epsilon}), \end{aligned} \quad (12)$$

where dx assigns measure 1 to $(R/\mathbb{Z}_p)^{\text{Prim}}$, and additionally,

$$c_2^{(i)} = \begin{cases} \frac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{30\Gamma(2/3)} & \text{if } i = 0, \\ \frac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{10\Gamma(2/3)} & \text{if } i = 1; \end{cases} \quad \text{and} \quad \Sigma_p = \begin{cases} \Sigma_p^{\text{ntr}} & \text{if } p \notin S, \\ \Sigma_p^{\text{tr}} & \text{if } p \in S. \end{cases}$$

(Recall that for any prime p , Σ_p^{tr} denotes the set of all isomorphism classes of maximal cubic orders over \mathbb{Z}_p that are totally ramified, and Σ_p^{ntr} denotes the set of all isomorphism classes of maximal cubic orders over \mathbb{Z}_p that are not totally ramified.)

In order to compute the second main term's constant, we combine Table 1, Lemma 28, and Lemma 37 in [2] to determine

$$\sum_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{\#\text{Aut}(R)} \int_{(R/\mathbb{Z})^{\text{Prim}}} i(x)^{2/3} dx = \begin{cases} \frac{1}{p(p+1)} + \frac{1}{p^{4/3}(p+1)} & \text{if } p \in S, \text{ and} \\ \frac{p^{1/3}}{p^{1/3}-1} - \frac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)} & \text{if } p \notin S. \end{cases}$$

We then calculate that $\prod_p (1 - p^{-1/3}) \cdot \prod_{R \in \Sigma_p} \frac{1}{\text{Disc}_p(R)} \cdot \frac{1}{\#\text{Aut}(R)} \int_{(R/\mathbb{Z})^{\text{Prim}}} i(x)^{2/3} dx$ is equal to

$$\begin{aligned} & \prod_p (1 - p^{-1/3}) \left(\frac{p^{1/3}}{p^{1/3}-1} - \frac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)} \right) \cdot \prod_{p \in S} \frac{\frac{1}{p(p+1)} + \frac{1}{p^{4/3}(p+1)}}{\frac{p^{1/3}}{p^{1/3}-1} - \frac{p^{2/3}+p^{1/3}}{p(p+1)(p^{1/3}-1)}} \\ &= \prod_p (1 - p^{-1/3}) \cdot \left(\frac{p^{1/3}p(p+1) - p^{2/3} - p^{1/3}}{p(p+1)(p^{1/3}-1)} \right) \cdot \prod_{p \in S} \frac{p^{2/3}-1}{p^{8/3}+p^{5/3}-p-p^{2/3}} \\ &= \prod_p 1 - \frac{p^{1/3}+1}{p(p+1)} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \cdot \frac{1-p^{-2/3}}{1 - \frac{p^{1/3}+1}{p(p+1)}}. \end{aligned}$$

We can thus conclude the following refinement of Theorem 3.2.

Theorem 5.1. *Let S denote a set of primes not containing 3, and let $n_i = |\text{Aut}(\mathbb{R}^{3-2i} \oplus \mathbb{C}^i)|$ for $i = 0$ or 1 . Let \mathcal{K}_S denote the set of isomorphism classes of cubic fields that are totally ramified exactly at the primes $p \in S$.*

$$\begin{aligned} N_3^{(i)}(\mathcal{K}_S, X) &= \frac{3}{n_i \pi^2} \cdot \prod_{p \in S} \frac{1}{p(p+1)} \cdot X \\ &+ \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3}+1}{p(p+1)} \right) \prod_{p \in S} \left(\frac{1}{p(p+1)} \cdot \frac{1-p^{-2/3}}{1 - \frac{p^{1/3}+1}{p(p+1)}} \right) \cdot X^{5/6} \\ &+ O_\epsilon(X^{5/6-1/48+\epsilon}), \end{aligned}$$

where

$$c_2^{(i)} = \begin{cases} \frac{\sqrt{3}\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{30\Gamma(2/3)} & \text{if } i = 0, \text{ and} \\ \frac{\zeta(2/3)\Gamma(1/3)(2\pi)^{1/3}}{10\Gamma(2/3)} & \text{if } i = 1. \end{cases}$$

We are now ready to prove Theorem 4. Let c be a positive integer coprime to 3, and let S_c denote the set of primes dividing c . Proposition 2.9 combined with Theorem 5.1 implies that

$$\begin{aligned}
\sum_{0 < (-1)^i \text{Disc}(K_2) < X} \# \text{Cl}_3^-(K_2, c) &= 1 + 2 \cdot \sum_{S \subseteq S_c} N_3^{(i)}(\mathcal{K}_S, X \cdot \prod_{p \in S} p^2) \\
&= 1 + 2 \cdot \left[\frac{3}{n_i \pi^2} \cdot \sum_{S \subseteq S_c} \left(\prod_{p \in S} \frac{1}{p(p+1)} \cdot X \cdot \prod_{p \in S} p^2 \right. \right. \\
&\quad \left. \left. + \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)} \right) \cdot \prod_{p \in S} \left(\frac{1}{p(p+1)} \cdot \frac{1 - p^{-2/3}}{1 - \frac{p^{1/3} + 1}{p(p+1)}} \right) \cdot X^{5/6} \cdot \prod_{p \in S} p^{5/3} \right. \right. \\
&\quad \left. \left. + O_\epsilon(X^{5/6 - 1/48 + \epsilon}) \right) \right].
\end{aligned}$$

Simplifying, we conclude

$$\begin{aligned}
\sum_{0 < (-1)^i \text{Disc}(K_2) < X} \# \text{Cl}_3^-(K_2, c) &= 1 + 2 \cdot \left[\frac{1}{n_i} \cdot \prod_{p \in S} \left(1 + \frac{p}{p+1} \right) \cdot \sum_{0 < (-1)^i \text{Disc}(K_2) < X} 1 \right. \\
&\quad \left. + \frac{c_2^{(i)}}{\zeta(2)} \cdot \prod_p \left(1 - \frac{p^{1/3} + 1}{p(p+1)} \right) \cdot \prod_{p \in S} \left(1 + \frac{p(1 - p^{1/3})}{1 - \frac{p^{1/3} + 1}{p(p+1)}} \right) \cdot X^{5/6} \right] \\
&\quad + O_\epsilon(X^{5/6 - 1/48 + \epsilon}).
\end{aligned}$$

Combining with Proposition 3.1 and (5), we deduce Theorem 4.

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